MINIMIZING THE MASS OF A CYLINDRICAL—LAMINAR THERMOPROTECTIVE SHELL

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UDC 539.4:678+518.5

The problem of minimizing the mass of heat insulation of a laminar cylinder, while ensuring the specified damping of external temperature perturbations, is considered.

In [1, 2], the problem of minimizing the mass and thickness of a plane—laminar thermoprotective panel, while quenching the harmonic temperature perturbations to a specified level, was solved. The same problem was solved for nonharmonic perturbations in [3]. In [4], the optimal structure of a plane panel of constant thickness providing maximal quenching of the harmonic temperature perturbation was found.

In the present work, the problem considered in [1-3] is solved, with some additional peculiarities, for the case of cylindrical symmetry.

Suppose that a three-layer cylinder (Fig. 1) is placed in a medium with a periodically varying temperature $T_0(t)$. The heat transfer at the external boundary of the cylinder $r = R_2$ is described by a boundary condition of the third kind. When $r = 0$, the condition of finiteness of the temperature is satisfied. The contact between the layers is supposed to be ideal. The radius of the internal layer and the thickness of the third are fixed. The structure of the external and internal layers is specified (the thermophysical properties in these layers may be constant, or depend continuously or piecewise-continuously on the radius). The structure and external radius of the middle layer, occupying the region $r \in [R_1, l]$, is not fixed in advance. This layer may be synthesized from a specified finite set of materials. It is required to find the structure of the middle layer and the value of l such that the total mass of the whole cylinder is a minimum and the temperature perturbation when $r = R_1$ at the boundary of the first and second layers reaches a specified value.

In each layer with continuously varying properties, the temperature satisfies the heat-conduction equation in cylindrical coordinates

$$c(r)\frac{\partial T(r,t)}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\lambda(r)\frac{\partial T(r,t)}{\partial r}\right), \tag{1}$$

the boundary conditions

$$\lambda\frac{\partial T(r,t)}{\partial r} = 0,\ \ \lambda\frac{\partial T(r,t)}{\partial r} = \alpha[T_0(t) - T(R_2, t)], \tag{2}$$

the ideal-contact conditions

$$[T(r,\ t)]_-^+ = \left[\lambda\frac{\partial T(r,t)}{\partial r}\right]_-^+ = 0,\ r \in [0,\ R_2] \tag{3}$$

and the periodicity condition

$$T(r, t + t^*) = T(r,\ t). \tag{4}$$

The temperature of the external medium $T_0(t)$, which is a periodic function of the time, is written in the form of a Fourier series

$$T_0(t) = \mathrm{Re}\sum_{k=0}^{\infty} z_k \cdot \exp(ik\omega t), \tag{5}$$

Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 53, No. 4, pp. 636-642, October, 1987.

 0022-0841/87/5304-1194$12.50

where $\omega = 2\pi/t^*$; z_k is the spectral density of the external temperature perturbation.

The condition in Eq. (4) means that steady conditions are considered, i.e., the temperature and heat flux in each layer may be written in the form

$$T(r, t) = \mathrm{Re} \sum_{k=0}^{\infty} y_{1k}(r) \exp(ik\omega t),$$

$$\lambda \frac{\partial T(r, t)}{\partial r} = \mathrm{Re} \sum_{k=0}^{\infty} y_{2k}(r) \exp(ik\omega t),$$

where y_{1k}, y_{2k} are the complex amplitudes of the temperature and heat flux, respectively, at frequency $k\omega$. The ideal-contact condition in Eq. (3) implies the continuity of $y_k(r) = \{y_{1k}, y_{2k}\}$ for all $r \in [0, R_2]$. At each k, these amplitudes satisfy the system of equations

$$y'_k(r) = A_k y_k(r), \quad (6)$$

$$A_k = \begin{bmatrix} 0 & 1/r \\ ik\omega c & -1/r \end{bmatrix},$$

which is valid over the whole interval $r \in [0, R_2]$, and the boundary conditions

$$y_{2k}(0) = 0, \quad y_{2k}(R_2) = \alpha (z_k - y_{1k}(R_2)). \quad (7)$$

In formulating the optimization problem, $y_{1k}(r)$, $y_{2k}(r)$ play the role of phase variables, and the system in Eqs. (6) and (7) plays the role of the system being controlled [5].

The structure and external radius of the second layer l are chosen as the control. It is expedient to introduce the characteristic function of the laminar medium u(r), which, at each point $r \in [R_1, l]$ takes an integer value equal to the ordinal number of the material at this point in the initial set. It is evident that specifying u(r) uniquely determines the number, dimensions, and material of the layers, in other words, the structure of the laminar medium. In addition, u(r) uniquely determines the distribution function of all the thermophysical properties of the medium over the radius, i.e., $\lambda = \lambda[u(r)]$, $c = c[u(r)]$, $\rho = \rho[u(r)]$. Therefore, the pair $\{u(r), l\}$ is chosen as the controlling variables. Here the function u(r) belongs to the class of piecewise-constant functions

$$u(r) = \{j_s \mid r_s \leqslant r < r_{s+1}\}, \quad s = \overline{1, S}, \quad (8)$$

the region of values of which is a finite set consisting of integers from 1 to M

$$j_s \in \{\overline{1, M}\}, \quad (9)$$

where M is the number of materials of the specified set. Since the problem of minimizing the mass is posed, the minimizing functional takes the form

$$F_0[u(r), l] \equiv \int_{R_1}^{l} r\rho[u(r)]\, dr + \int_{l}^{R_2} r\rho(r)\, dr \to \min.$$

The density distribution when $r \in [l, R_2]$ is fixed, and hence the functional may be written in the form

$$F_0[u(r), l] = \int_{R_1}^{l} r\rho[u(r)]\, dr + \int_{0}^{L} (x + l)\, w(x)\, dx, \quad (10)$$

where $L = R_2 - l = \mathrm{const}$; $x = r - l$; the function w(x) is specified on the segment $x \in [0, L]$.

The temperature perturbation at point r is understood to be the quantity [3]

$$Q(r) = \int_{0}^{t^*} T^2(r, t)\, dt = \frac{t^*}{2} \sum_{k=1}^{\infty} |y_{1k}(r)|^2. \quad (11)$$

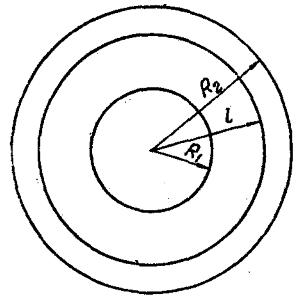

Fig. 1. Structure of laminar cylinder.

Taking account of Eq. (11), the constraint imposed on the temperature perturbation at the point $r = R_1$ is written in the form of a functional

$$F_1[u(r),\ l] \equiv \sum_{k=1}^{\infty} |y_{1k}(R_1)|^2 - \eta^2 \sum_{k=1}^{\infty} |z_k|^2 = 0,\ \eta^2 < 1. \tag{12}$$

Here the first sum describes the temperature perturbation at the point $r = R_1$, and the second describes the temperature perturbation at points of the external medium; the parameter η is specified in advance.

Mathematically, the optimization problem may be formulated as follows. Among the piecewise-constant functions $u(r)$ in Eqs. (8) and (9) defined on the segment $[R_1, l]$ and the numbers $l > R_1$, it is required to find the pair $\{u^\circ(r), l^\circ\}$ corresponding to a minimum of the functional in Eq. (10) under the constraint in Eq. (12). The phase variables appearing in Eq. (12) are determined from the solution of the boundary problem in Eqs. (6) and (7). This optimization problem differs from those considered in [1-3] in two respects. First, the functional in Eq. (12) is determined at the internal point of the segment $[0, R_2]$ at which the boundary problem for the phase variables is specified. This means that the vector of conjugate variables at the point $r = R_1$ will undergo a discontinuity [5]. Second, the control $u(r)$ only varies on the internal section $[R_1, l]$ of the segment $[0, R_2]$. These features are easily taken into account in deriving the necessary conditions for optimality (NCO), but it is simplest to reduce the problem to that studied in [1-3], transferring the boundary conditions from the point $r = 0$ to the point $r = R_1$ and from the point $r = R_2$ to the point $r = l$. The solution of Eq. (6) is written in the form

$$y_{2k}(r) = p_k(r)\, y_{1k}(r) + q_k(r).$$

The fitting factor $p_k(r)$ is found from the solution of the Riccati equation

$$p_k' + \frac{1}{\lambda} p_k^2 + \frac{1}{r} p_k - ik\omega c = 0,\ p_k(R_2) = -\alpha,$$

and $q_k(r)$ from the linear equation

$$q_k' + q_k\left(\frac{p_k}{\lambda} + \frac{1}{r}\right) = 0,\ q_k(R_2) = \alpha z_k.$$

Then the boundary conditions at $r = R_1$ and $r = l$ analogous to Eq. (7) may be found

$$y_{2k}(R_1) = \alpha_{in} y_{1k}(R_1),\ y_{2k}(l) = \alpha_{ex}(y_{1k}(l) - \hat{z}_k), \tag{13}$$

where $\alpha_{in} = p_k(R_1)$; $\alpha_{ex} = p_k(l)$; $\hat{z}_k = q_k(l)/p_k(l)$.

Thus, in the above mathematical formulation of the optimization problem, the phase variables $y_{1k}(R_1)$ are found from the solution of the boundary problem in Eqs. (6) and (13), and the optimization problem itself coincides in form with those considered in [1-3].

Since the region of values of the controlling function in Eq. (8) is the discrete set in Eq. (9), finite variations in sets of smaller measure are used in deriving the NCO and organizing the computational algorithm. Analogously to [1,4], the NCO for the given problem may be formulated as follows. Suppose that the pair $\{u^\circ(r), l^\circ\}$ is the optimal control, and y_{1k}, y_{2k} define the corresponding phase trajectory determined at each k as the solution of the boundary problem in Eqs. (6) and (13) such as to satisfy the constraint in Eq. (12). Then there exist vector functions $\psi = \{\psi_{1k}, \psi_{2k}\}$ and $\varphi_k = \{\varphi_{1k}, \varphi_{2k}\}$, satisfying the boundary problems.

$$\psi_k' = -A_k^* \psi_k,$$
$$\psi_{1k}(R_1) = -\alpha_{in}\psi_{2k}(R_1) - 2\bar{y}_{1k}(R_1), \quad \psi_{1k}(l) = -\alpha_{ex}\psi_{2k}(l), \tag{14}$$

and

$$\varphi_k' = A_k \varphi_k,$$
$$\varphi_{2k}(R_1) = \alpha_{in}\varphi_{1k}(R_1), \tag{15}$$
$$\varphi_{2k}(l) = \alpha_{ex}\varphi_{1k}(l) - ik\omega c[u(l)]\, y_{1k}(l) + \left[\frac{1}{l} + \frac{\alpha_{ex}}{\lambda[u(l)]}\right] y_{2k}(l)$$

and such that the Hamiltonian constructed on this basis

$$H(y_k, \psi_k, \varphi_k, r, n) = -r\rho(n) + B^{-1}\left\{l\rho[u(l)] + \int_0^L w(x)\,dx\right\} \mathrm{Re} \sum_{k=1}^{\infty} \langle \psi_k, A_k(n) y_k \rangle, \tag{16}$$
$$B = 2\mathrm{Re} \sum_{k=1}^{\infty} y_{1k}(R_1)\, \varphi_{1k}(R_1) = \mathrm{const}$$

reaches its maximum relative to the argument n at the optimal control at almost all $r \in [R_1, l]$ (the expression $\langle a, b \rangle$ denotes a scalar product)

$$H(\cdot, u^0) = \max_{n \in \overline{1,M}} H(\cdot, n).$$

In investigating the NCO, it is expedient to write the Hamiltonian H in the form

$$H(\cdot, n) = -\beta_1(\cdot)\rho(n) + \beta_2(\cdot)\frac{1}{\lambda(n)} + \beta_3(\cdot)c(n) - \beta_4(\cdot), \tag{17}$$
$$\beta_1(\cdot) = r, \quad \beta_2(\cdot) = B^{-1}\left(l\rho[u(l)] + \int_0^L w(x)\,dx\right) \mathrm{Re} \sum_{k=1}^{\infty} y_{2k}\psi_{1k},$$
$$\beta_3(\cdot) = B^{-1}\left(l\rho[u(l)] + \int_0^L w(x)\,dx\right) \mathrm{Re} \sum_{k=1}^{\infty} ik\omega y_{1k}\psi_{2k},$$
$$\beta_4(\cdot) = \frac{1}{r} B^{-1}\left(l\rho[u(l)] + \int_0^L w(x)\,dx\right) \mathrm{Re} \sum_{k=1}^{\infty} y_{2k}\psi_{2k},$$

where $(\cdot)$ denotes the set of arguments y_k, ψ_k, φ_k, r of the Hamiltonian, calculated for the optimal control; β_1, β_2, β_3, β_4 are functions continuous in r. Inside the interval $[R_1, l]$, complex amplitudes of the temperature y_{1k} and heat flux y_{2k} cannot vanish at any k inside the interval $[R_1, l]$. An analogous statement holds for ψ_k. Taking this into account, it may be shown that all the β_i in Eq. (17) are larger than zero. The condition $\beta_i > 0$ may be used for preliminary selection of those materials from the initial set which may be used in the optimal construction. Consider the function

$$G(\gamma_1, \gamma_2, n) = -\rho(n)\sin\gamma_1 + \frac{1}{\lambda(n)}\cos\gamma_1\cos\gamma_2 + c(n)\cos\gamma_1\sin\gamma_2 \tag{18}$$

and the set

$$D = \left\{\arg\max_{n \in \{1,M\}} G(\gamma_1, \gamma_2, n) \,\middle|\, \gamma_1 \in \left[0, \frac{\pi}{2}\right], \ \gamma_2 \in \left[0, \frac{\pi}{2}\right]\right\},$$

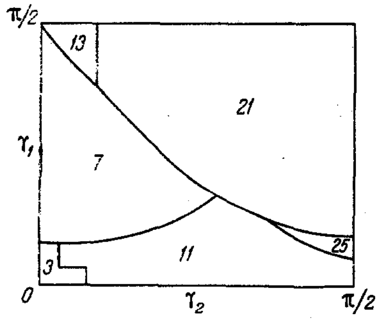

Fig. 2. Diagram of materials appearing in the optimal construction and their mutual position.

TABLE 1. Thermophysical Properties of Materials

No. of material	Density, kg/m³	Thermal conduct., J/c·m·K	Volume specific heat, 10⁶ J/m³·K	No. of material	Density, kg/m³	Thermal conduct., J/c·m·K	Volume specific heat, 10⁶ J/m³·K
1	150	0,047	0,195	20	50	0,048	0,04
2	100	0,041	0,13	21	1000	0,15	2,3
3	40	0,038	0,052	22	800	0,13	1,84
4	125	0,052	0,163	23	600	0,1	1,38
5	600	0,116	1,381	24	400	0,081	0,92
6	80	0,041	0,12	25	200	0,058	0,46
7	60	0,035	0,09	26	800	0,16	1,84
8	100	0,047	0,17	27	600	0,12	1,38
9	75	0,043	0,128	28	800	0,163	1,831
10	50	0,041	0,085	29	300	0,07	0,69
11	40	0,038	0,068	30	1000	0,29	0,84
12	200	0,041	0,2	31	800	0,21	0,672
13	100	0,035	0,1	32	600	0,14	0,504
14	300	0,076	0,3	33	400	0,1	0,336
15	200	0,064	0,2	34	300	0,081	0,252
16	350	0,091	0,28	35	1000	0,23	0,84
17	300	0,084	0,24	36	800	0,17	0,672
18	200	0,07	0,16	37	600	0,104	1,368
19	100	0,056	0,08				

composed of the numbers of materials in the initial set which permit a maximum of the function G with respect to the argument n, when γ_1 and γ_2 independently take values in the range from 0 to $\pi/2$. It may be asserted that the materials which may appear in the minimum-mass construction have numbers from the set D. The other materials may be excluded from consideration in advance. In Fig. 2, the numbers of the materials from Table 1 which permit a maximum of the function in Eq. (18) with variation in γ_1 and γ_2 within the corresponding ranges are shown. It is evident from Fig. 2 that only six of the 37 initial materials may be used in the optimal construction. In addition, it may be asserted that, in the optimal construction, materials with numbers corresponding to regions with common boundaries in Fig. 2 may be side by side. Thus, a preliminary idea of the structure of the optimal construction may be obtained.

The results of numerical calculations are now considered for the following parameters of the problem. The temperature of the external medium varies according to a sinusoidal law with a period equal to one day. The external heat-transfer coefficient $\alpha = 20$ J/m²·sec·K. The internal layer of the cylinder has a radius $R_1 = 0.6$ m and consists of material with the following thermophysical properties: $\rho = 500$ kg/m³, $\lambda = 0.15$ J/m·sec·K, $c = 0.847 \cdot 10^6$ J/m³·K. The external layer (thickness 0.02 m) consists of material with $\rho = 1600$ kg/m³, $\lambda = 0.65$ J/m·sec·K, $c = 2.6 \cdot 10^6$ J/m³·K. The materials of the initial set from which the internal layer may be formed are given in Table 1. When η in Eq. (12) is 1/10, which corresponds to tenfold damping of the external temperature perturbation, the following construction is optimal: the region from 0.6 to 0.865 m corresponds to material 11 and the region from 0.865 to 0.88 m to material 3.

NOTATION

t, time; r, radius; ρ, density; c, volume specific heat; λ, thermal conductivity; α, external heat-transfer coefficient; T, temperature; t*, period; ω, frequency; η, specified constant.

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PERIODIC PULSE HEATING OF METALS

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UDC 536.241:535.211

The article deals with the effect of the time-dependent form and heating by preceding pulses on the resulting temperature field in metal subjected to periodic pulsed heat treatment.

The widespread use of lasers in various technological processes requires the construction of theoretical models describing the effect of laser radiation on substances. At present there are available many theoretical investigations dealing with continuous and pulsed treatment of materials; their results were generalized, e.g., in [1-7]. However, periodic pulsed loading received much less attention [8, 9] although in practice it is ever more widely used.

A unidimensional temperature field T(x, t) in a half-space with an arbitrary time dependence of the energy flux density can be written in the form [10]

$$T = T_n + \frac{1}{\lambda}\sqrt{\frac{a}{\pi}}\int_0^t q(\xi)\,\frac{\exp\left[-x^2/4a(t-\xi)\right]}{\sqrt{t-\xi}}\,d\xi. \tag{1}$$

For expression (1) we use the Laplace–Carson transformation [11]

$$\bar{T} = T_n + \frac{\sqrt{a}}{\lambda}\,\frac{\bar{q}(p)}{\sqrt{p}}\exp\left(-x\sqrt{p/a}\right). \tag{2}$$

To find $\bar{q}(p)$, we expand q(t) into a Fourier series, Eq. (2) assumes the form

$$\bar{T} = T_n + \frac{\sqrt{a}}{\lambda}\sum_{k=-\infty}^{+\infty} c_k\,\frac{\sqrt{p}\exp\left(-x\sqrt{p/a}\right)}{p-i\omega_k},$$

where

$$c_k = \frac{1}{\tau}\int_0^\tau q(t)\exp(-i\omega_k t)\,dt,\ \ \omega_k = k\omega_0,\ \ k = 0,\ 1,\ \ldots,$$

$\omega_0 = 2\pi/\tau$, τ is the period of action, i.e., the interval between the instants of the onset of two adjacent pulses. Going over to the original, we have

$$T = T_n + \frac{\sqrt{a}}{\lambda}\sum_{k=-\infty}^{+\infty} c_k\,\frac{(i-1)\exp(i\omega_k t)}{2\sqrt{2\omega_k}}\left\{\exp\left[\frac{x}{2}\sqrt{\frac{2\omega_k}{a}}(i+1)\right]\times\right.$$

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